8 Unendliche Reihen, Potenzreihen, Taylor-Reihen, Fourier-Reihen

8.1.1 Grundlegende Definitionen und Eigenschaften unendlicher Reihen

1)

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2\cdot n-1)\cdot(2\cdot n+1)}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\cdot(n+1)}$

c) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\cdot(n+1)}$

d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(3\cdot n-2)\cdot(3\cdot n+1)}$

e) $\displaystyle\sum_{n=2}^{\infty} \frac{4}{n\cdot(n+4)}$

f) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2\cdot n-1}+\frac{1}{2\cdot n}\right)\cdot\frac{1}{n+2}$

2)

a) $S=\frac{8}{9}$ b) $S=\frac{10}{7}$ c) $S=\frac{12}{5}$

d) $S=\frac{3}{2}$ e) $S=2$ f) $S=\frac{2}{3}$

8.1.3.1 Quotientenkriterium

1)

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\cdot n-1}$ $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$

b) $\displaystyle\left[\sum_{n=1}^{\infty} 1+\frac{1}{10^n}\right]$ $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$

c) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n}$ $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{1}{3}$

d) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n+n^2}$ $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{1}{3}$

e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$

f) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$

g) $\sum_{n=1}^{\infty} \frac{3 \cdot n - 2}{n!}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$

h) $\sum_{n=1}^{\infty} \frac{5^n}{12 \cdot n^2}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 5$

i) $\sum_{n=1}^{\infty} \frac{n}{2^n}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{2}$

j) $\sum_{n=1}^{\infty} \frac{5 \cdot n - 2}{5 \cdot n + 2}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$

k) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$

l) $\sum_{n=1}^{\infty} \frac{2^n}{n+1}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 2$

2)

a) $\sum_{n=0}^{\infty} \frac{10^n}{n!}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$

b) $\sum_{n=0}^{\infty} \frac{1}{(2 \cdot n + 1) \cdot 2^{2 \cdot n + 1}}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{4}$

c) $\sum_{n=0}^{\infty} \frac{(2 \cdot n + 1)}{2^{n+1}}$ oder $\sum_{n=1}^{\infty} \frac{(2 \cdot n - 1)}{2^n}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{2}$

d) $\sum_{n=1}^{\infty} \frac{\ln(2)^n}{n!}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$

e) $\sum_{n=1}^{\infty} \frac{1}{10^n + 1}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{10}$

f) $\sum_{n=1}^{\infty} \frac{n}{5^n}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{5}$

g) $\sum_{n=0}^{\infty} \frac{1}{2^{2\cdot n}}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{4}$

h) $\sum_{n=1}^{\infty} n\cdot\left(\frac{1}{2}\right)^{n-1}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{2}$

i) $\sum_{n=1}^{\infty} (-1)^{n+1}\cdot\frac{2^n}{n}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 2$

j) $\sum_{n=1}^{\infty} \frac{3^{2\cdot n}}{2\cdot n!}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$

k) $\sum_{n=1}^{\infty} n^3\cdot e^{(-1)^n\cdot n}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \text{nicht}\cdot\text{definiert}$

8.1.3.2 *LEIBNIZsches Konvergenzkriterium für alternierende Reihen*

1)

a) $\sum_{n=1}^{\infty} (-1)^{n+1}\cdot\frac{1}{n}$ $\lim_{n \to \infty} a_n = 0$

b) $\sum_{n=1}^{\infty} (-1)^{n+1}\cdot\frac{2\cdot n+1}{3\cdot n}$ $\lim_{n \to \infty} a_n = \frac{2}{3}$

c) $\sum_{n=1}^{\infty} (-1)^{n+1}\cdot\frac{1}{n!}$ $\lim_{n \to \infty} a_n = 0$

d) $\sum_{n=0}^{\infty} (-1)^{n}\cdot\frac{1}{2\cdot n-1}$ $\lim_{n \to \infty} a_n = 0$

e) $\sum_{n=1}^{\infty} (-1)^{n+1}\cdot\frac{1}{3^{n+1}}$ $\lim_{n \to \infty} a_n = 0$

f) $$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{3 \cdot n - 1}{4 \cdot n}$$ $$\lim_{n \to \infty} a_n = \frac{3}{4}$$

g) $$\left[\sum_{n=2}^{\infty} \frac{1}{n^2} - \frac{1}{2^n}\right]$$ $$\lim_{n \to \infty} a_n = 0$$

h) $$\sum_{n=1}^{\infty} (-1)^n \cdot \left(\frac{1}{n} + n\right)$$ $$\lim_{n \to \infty} a_n = \infty$$

2)

a) $$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n!}$$ $$\lim_{n \to \infty} a_n = 0$$

b) $$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(2 \cdot n + 1)}$$ $$\lim_{n \to \infty} a_n = 0$$

c) $$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n^2}$$ $$\lim_{n \to \infty} a_n = 0$$

d) $$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n \cdot 5^{2 \cdot n - 1}}$$ $$\lim_{n \to \infty} a_n = 0$$

e) $$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{2^n}{n^2}$$ $$\lim_{n \to \infty} a_n = \infty$$

8.2 *Potenzreihen*

1)

a) $\sum_{n=0}^{\infty} \frac{n}{n+1} \cdot x^n$ $\qquad \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1$

b) $\sum_{n=0}^{\infty} \frac{2^n \cdot (n!)^2 \cdot (n+1)}{(2 \cdot n+1)!} \cdot x^n$ $\qquad \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 2$

c) $\sum_{n=0}^{\infty} \frac{2^n}{n+1} \cdot x^n$ $\qquad \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{2}$

d) $\sum_{n=1}^{\infty} n^n \cdot x^n$ $\qquad \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 0$

e) $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2 \cdot n)!} \cdot x^n$ $\qquad \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 4$

f) $\sum_{n=0}^{\infty} \frac{10^n}{n!} \cdot x^n$ $\qquad \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \infty$

g) $\sum_{n=0}^{\infty} \frac{(2 \cdot n+1)!}{2^n \cdot (n!)^2} \cdot x^n$ $\qquad \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{2}$

h) $\sum_{n=0}^{\infty} \left(2^n - n\right) \cdot x^n$ $\qquad \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{2}$

i) $\sum_{n=0}^{\infty} (2 \cdot n - 1) \cdot x^n$ $\qquad \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1$

j) $\sum_{n=0}^{\infty} \left(\frac{x^{-1}}{2} \right)^n$ $\qquad \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 2$

2)

a) $\sum_{n=1}^{\infty} n \cdot x^n$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1 \quad |x| < 1$

b) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^n}{n}$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1 \quad |x| < 1$

c) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1 \quad |x| < 1$

d) $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 2 \quad |x| < 2$

e) $\sum_{n=0}^{\infty} x^n \cdot (2 \cdot n - 1)^2$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1 \quad |x| < 1$

f) $\sum_{n=0}^{\infty} \frac{x^n}{n^2 + n!}$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \infty \quad |x| = \infty$

g) $\sum_{n=1}^{\infty} x^{n-2} \cdot (2 \cdot n - 3)^{n-1}$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 0 \quad x = 0$

h) $\sum_{n=1}^{\infty} \frac{(1 - x)^n}{n}$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1 \quad 0 < x < 2$

i) $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{(x - 1)^n}{n}$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1 \quad 0 < x < 2$

j) $\sum_{n=1}^{\infty} \left(\frac{x^n}{n \cdot \pi} \right)^{(-1)^n}$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1 \quad |x| < 1$

8.3.1 *Mac LAURINsche Reihe*

a) $f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^{2 \cdot n}}{n}$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1$ $\quad |x| < 1$

b) $f(x) = \sum_{n=0}^{\infty} \frac{x^{2 \cdot n}}{(2 \cdot n)!}$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \infty$ $\quad |x| = \infty$

c) $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \infty$ $\quad |x| = \infty$

d) $f(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^n}{n!}$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \infty$ $\quad |x| = \infty$

e) $f(x) = \sum_{n=0}^{\infty} (-1)^n \cdot x^n$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1$ $\quad |x| < 1$

f) $f(x) = \sum_{n=0}^{\infty} x^n$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1$ $\quad |x| < 1$

g) $f(x) = \sum_{n=1}^{\infty} \left(\frac{\pi \cdot x}{4} \right)^{2 \cdot n - 1} \cdot \frac{(-1)^{n-1}}{(2 \cdot n - 1)!}$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \infty$ $\quad |x| = \infty$

h) $f(x) = \sum_{n=0}^{\infty} \frac{(x \cdot \ln(2))^n}{n!}$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \infty$ $\quad |x| = \infty$

i) $f(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{n+1}}{n!}$ $\quad r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \infty$ $\quad |x| = \infty$

8.3.2 TAYLOR*sche Reihe*

a) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{(x-1)^n}{n}$ $r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1$ $0<x<2$ $x_0 = 1$

b) $\sum_{n=0}^{\infty} (-1)^n \cdot (x-1)^n$ $r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1$ $0<x<2$ $x_0 = 1$

c) $\frac{1}{2} \cdot (x+1) + \sum_{n=3}^{\infty} -\frac{(2 \cdot n - 5)! \cdot (1-x)^{n-1}}{4^{n-2} \cdot (n-3)! \cdot (n-1)!}$

$r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1$ $0<x<2$ $x_0 = 1$

d) $\sum_{n=1}^{\infty} \frac{-(2-x)^n}{n}$ $r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1$ $1<x<3$ $x_0 = 2$

e) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{(x-1)^{n-1}}{n}$ $r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1$ $0<x<2$ $x_0 = 1$

f) $4 + \sum_{n=1}^{\infty} (n+3) \cdot (1-x)^n$ $r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1$ $0<x<2$ $x_0 = 1$

8.3.4 *Integration durch Potenzreihenentwicklung*

a) $\int \frac{\sin(x)}{x} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2 \cdot n + 1}}{(2 \cdot n + 1) \cdot (2 \cdot n + 1)!}$

b) $\int e^{x^3} \, dx = \sum_{n=0}^{\infty} \frac{x^{3 \cdot n + 1}}{n! \cdot (3 \cdot n + 1)}$ c) $\int \frac{1}{1 - x^2} \, dx = \sum_{n=0}^{\infty} \frac{x^{2 \cdot n + 1}}{2 \cdot n + 1}$

d) $\int \ln\left(\frac{1+x}{1-x}\right) dx = \sum_{n=1}^{\infty} \frac{x^{2 \cdot n}}{(2 \cdot n - 1) \cdot n}$