9 Differentialrechnung für Funktionen von mehreren Variablen

9.2 Partielle Differentiation

1)

a) $z_x = 2\cdot x\cdot y^3 - 4\cdot y^2 - 6$ $\quad z_y = 3\cdot x^2\cdot y^2 - 8\cdot x\cdot y + 5$

b) $z_x = \frac{1}{\sqrt{y^2 - x^2}}$ $\quad z_y = \frac{-x}{y\cdot\sqrt{y^2 - x^2}}$

c) $z_x = \frac{1}{2}\cdot\sqrt{x^y}\cdot\frac{y}{x}$ $\quad z_y = \frac{1}{2}\cdot\sqrt{x^y}\cdot\ln(x)$

d) $z_x = -\frac{4\cdot y\cdot x}{\left(x^2 - y^2\right)^2}\cdot\left[1 + \tan\left[\frac{y}{\left(x^2 - y^2\right)}\right]^2\right]\cdot\tan\left[\frac{y}{\left(x^2 - y^2\right)}\right]$

$z_y = 2\cdot\frac{\left(x^2 + y^2\right)}{\left(x^2 - y^2\right)^2}\cdot\left[1 + \tan\left[\frac{y}{\left(x^2 - y^2\right)}\right]^2\right]\cdot\tan\left[\frac{y}{\left(x^2 - y^2\right)}\right]$

2)

a) $f_x = 2\cdot x\cdot\sin(y) - y^2\cdot\sin(x) + z^2\cdot\cos(x)\cdot\cos(y)$

$f_y = x^2\cdot\cos(y) + 2\cdot y\cdot\cos(x) - z^2\cdot\sin(x)\cdot\sin(y)$

$f_z = 2\cdot z\cdot\sin(x)\cdot\cos(y)$

b) $f_t = e^t\cdot\ln\left(x^2 + y^2\right)$ $\quad f_x = \frac{2\cdot x\cdot e^t}{x^2 + y^2}$ $\quad f_y = \frac{2\cdot y\cdot e^t}{x^2 + y^2}$

c) $f_t = e^{x\cdot z}$ $\quad f_x = z\cdot t\cdot e^{x\cdot z}$ $\quad f_z = x\cdot t\cdot e^{x\cdot z}$

d) $f_t = \frac{1}{x^2 + t}$ $\quad f_x = \frac{2\cdot x}{x^2 + t}$

e) $f_x = \frac{2\cdot x}{\sqrt{2\cdot x^2 + y}}$ $\quad f_y = \frac{1}{2\cdot\sqrt{2\cdot x^2 + y}}$

f) $u_x = (1 + x\cdot y)\cdot e^{x\cdot y}$ $\quad u_y = x^2\cdot e^{x\cdot y}$

g) $u_x = 2\cdot y$ $\quad u_y = 2\cdot x + \cos(y)$ $\quad u_z = 2\cdot z$

h) $$f_t = \frac{(-x\cdot\sin(t) + 2\cdot\sin(x))}{(x\cdot\sin(t) - 2\cdot t\cdot\cos(x))} - \frac{(x\cdot\cos(t) + 2\cdot t\cdot\sin(x))}{(x\cdot\sin(t) - 2\cdot t\cdot\cos(x))^2}\cdot(x\cdot\cos(t) - 2\cdot\cos(x))$$

$$f_x = \frac{(\cos(t) + 2\cdot t\cdot\cos(x))}{(x\cdot\sin(t) - 2\cdot t\cdot\cos(x))} - \frac{(x\cdot\cos(t) + 2\cdot t\cdot\sin(x))}{(x\cdot\sin(t) - 2\cdot t\cdot\cos(x))^2}\cdot(\sin(t) + 2\cdot t\cdot\sin(x))$$

i) $T_t = \ln\left(\frac{x}{y}\right)$ $\quad T_x = \frac{t}{x}$ $\quad T_y = -\frac{t}{y}$

3) $u_x = \dfrac{\sqrt[3]{t}}{3\cdot x\cdot\left(\sqrt[3]{x} - \sqrt[3]{t}\right)}$ $\quad u_t = \dfrac{\sqrt[3]{x}}{3\cdot t\cdot\left(\sqrt[3]{t} - \sqrt[3]{x}\right)}$

9.2.2 *Partielle Ableitungen höherer Ordnung*

1)

a)

$f_x = 0$ $\quad f_y = 0$

$f_{xx} = 0$ $\quad f_{yy} = 0$ $\quad f_{xy} = f_{yx} = 0$

b) $f_x = \sin(y)$ $\quad f_y = x\cdot\cos(y)$

$f_{xx} = 0$ $\quad f_{yy} = -x\cdot\sin(y)$ $\quad f_{xy} = f_{yx} = \cos(y)$

c) $f_x = \dfrac{2\cdot x}{x^2 + y^2}$ $\quad f_y = \dfrac{2\cdot y}{x^2 + y^2}$

$f_{xx} = \dfrac{2}{x^2 + y^2} - \dfrac{2\cdot x}{x^2 + y^2}$ $\quad f_{yy} = \dfrac{2}{x^2 + y^2} - \dfrac{2\cdot y}{x^2 + y^2}$ $\quad f_{xy} = \dfrac{-4\cdot x\cdot y}{\left(x^2 + y^2\right)^2}$

d) $f_x = \dfrac{x}{\sqrt{x^2 + y^2}}$ $\quad f_y = \dfrac{y}{\sqrt{x^2 + y^2}}$

$f_{xx} = -\dfrac{x^2}{\sqrt{\left(x^2 + y^2\right)^3}}$ $\quad f_{yy} = -\dfrac{y^2}{\sqrt{\left(x^2 + y^2\right)^3}}$ $\quad f_{xy} = \dfrac{-x\cdot y}{\sqrt{\left(x^2 + y^2\right)^3}}$

e)

$$f_x = \cos(y)\cdot\cos(x\cdot\cos(y))\cdot e^{\sin(x\cdot\cos(y))}$$

$$f_y = -x\cdot\sin(y)\cdot\cos(x\cdot\cos(x))\cdot e^{\sin(x\cdot\cos(y))}$$

$$f_{xx} = e^{\sin(x\cdot\cos(y))}\cdot\cos(y)^2\cdot\left(\cos(x\cdot\cos(y))^2 - \sin(x\cdot\cos(y))\right)$$

$$f_{yy} = e^{\sin(x\cdot\cos(y))}\cdot\left[x^2\cdot\sin(y)^2\cdot\left(\cos(x\cdot\cos(y))^2 - \sin(x\cdot\cos(y))\right) - \cos(x\cdot\cos(y))\cdot x\cdot\cos(y)\right]$$

$$f_{xy} = e^{\sin(x\cdot\cos(y))}\cdot\sin(y)\cdot\left[x\cdot\cos(y)\cdot\left(\sin(x\cdot\cos(y)) - \cos(x\cdot\cos(y))^2\right) - \cos(x\cdot\cos(y))\right]$$

2)

a) $f_{xx} = \dfrac{-4\cdot x^2}{\sqrt{\left(2\cdot x^2 + y\right)^3}} + \dfrac{2}{\sqrt{2\cdot x^2 + y}}$ $\quad f_{yy} = \dfrac{-1}{4\cdot\sqrt{\left(2\cdot x^2 + y\right)^3}}$

$f_{xy} = -\dfrac{x}{\sqrt{\left(2\cdot x^2 + y\right)^3}}$

b) $T_{tt} = 0$ $\quad T_{xx} = -\dfrac{t}{x^2}$ $\quad T_{yy} = \dfrac{t}{y^2}$

$T_{tx} = \dfrac{1}{x}$ $\quad T_{ty} = -\dfrac{1}{y}$ $\quad T_{xy} = 0$

c) $u_{xx} = 2$ $\quad u_{yy} = e^y\cdot\ln(z)$ $\quad u_{zz} = -\dfrac{e^y}{z^2}$

$u_{xy} = 0$ $\quad u_{xz} = 0$ $\quad u_{yz} = \dfrac{e^y}{z}$

d) $w_{rr} = \dfrac{6\cdot t}{r^4}$ $\quad w_{tt} = 0$ $\quad w_{rt} = -\dfrac{2}{r^3}$

3) $f_{xy} = f_{yx} = \cot(x)$

9.2.3 *Das totale (vollständige) Differential*

1)

a) $df = 2\cdot\cos\left(x^2 + y^2\right)\cdot(x\cdot dx + y\cdot dy)$

b) $du = \frac{x}{2\cdot t}\cdot dt + \ln\left(\sqrt{t}\right)\cdot dx$

c) $du = 2\cdot y\cdot dx + (2\cdot x + \cos(y))\cdot dy + 2\cdot z\cdot dz$

d) $df = e^{x+y+z}\cdot(dt + t\cdot(dx + dy + dz))$

e) $dz = \left(3\cdot x^2\cdot y + y^3\right)\cdot dx + \left(x^3 + 3\cdot y^2\cdot x\right)\cdot dy$

f) $dz = \frac{1}{x + \sqrt{x^2 + y^2}}\cdot\left[\left(1 + \frac{x}{\sqrt{x^2 + y^2}}\right)\cdot dx + \frac{y\cdot dy}{\sqrt{x^2 + y^2}}\right]$

g) $dz = \frac{2\cdot y\cdot(y\cdot dx - x\cdot dy)}{\sqrt{x^2 + y^2}\cdot\left[\left(x + \sqrt{x^2 + y^2}\right)\cdot\left(-\sqrt{x^2 + y^2} + x\right)\right]}$

h) $dz = \cos\left(\frac{x}{y}\right)\cdot\cos\left(\frac{y}{x}\right)\cdot\left(\frac{dx}{y} - \frac{x\cdot dy}{y^2}\right) + \sin\left(\frac{x}{y}\right)\cdot\sin\left(\frac{y}{x}\right)\cdot\left(\frac{y\cdot dx}{x^2} - \frac{dy}{x}\right)$

2)

a) $dz = \left(12\cdot x^2\cdot y - 3\cdot e^y\right)\cdot dx + \left(4\cdot x^3 - 3\cdot x\cdot e^y\right)\cdot dy$

b) $dz = \frac{4\cdot t^2 + 2\cdot t}{(2\cdot t - 4\cdot x)^2}\cdot dx + \frac{2\cdot t^2 - 8\cdot t\cdot x - 2\cdot x}{(2\cdot t - 4\cdot x)^2}\cdot dt$

c) $dz = \frac{x^2 - 2\cdot x\cdot y - y^2}{(x - y)}\cdot dx + \frac{x^2 + 2\cdot x\cdot y - y^2}{(x - y)^2}\cdot dy$

d) $du = \frac{x}{x^2 + y^2 + z^2}\cdot dx + \frac{y}{x^2 + y^2 + z^2}\cdot dy + \frac{z}{x^2 + y^2 + z^2}\cdot dz$

9.2.3.1 *Partielle Ableitungen, wenn Funktionen in Parameterdarstellung vorliegen*

a) $\frac{du}{dt} = e^{\sin(t) - 2\cdot t^3}\cdot\cos(t) - 6\cdot t^2$

b) $\frac{du}{dt} = \left(2\cdot e^t + \sin(t)\right)\cdot e^t + \left(2\cdot\sin(t) + e^t\right)\cdot\cos(t)$

c) $\frac{dz}{du} = 3\cdot u^2\cdot\cos(v)\cdot\sin(v)\cdot(\cos(v) - \sin(v))$

$\frac{dz}{dv} = u^3\cdot(\sin(v) + \cos(v))\cdot(1 - 3\cdot\sin(v)\cdot\cos(v))$